

SINGULAR BEHAVIOUR OF ELECTRIC AND MAGNETIC FIELDS
IN DIELECTRIC MEDIA
IN A NON-LINEAR GRAVITATIONAL WAVE BACKGROUND

by

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Abstract Evolution of electric and magnetic fields in dielectric media, driven by the influence of a strong gravitational wave, is considered for four exactly integrable models. It is shown that the gravitational wave field gives rise to new effects and to singular behaviour in the electromagnetic field.

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1 Introduction

In 1970 Bocaletti et al opened up the discussion about the interaction of weak gravitational wave (GW) fields and static electromagnetic fields in vacuum [1]. In the seventies this problem has been studied in detail in the application to GW detection (see, e.g., [2]). The idea of these investigations was based on two suggestions (i) that the GW field is weak and (ii) the electromagnetic field is in vacuum. Item (ii) was motivated by the supposition that either the presence of matter or vacuum fluctuations make the velocities of both gravitational and electromagnetic waves different, therefore destroying the coherence between both types of waves [3] (see, also, [4]).

This problem has been taken up in [5] where items (i) and (ii) were dropped down and solutions in the presence of matter were considered for non-linear GW background. Specifically, an exact solvable model of evolution of an electromagnetic field in a GW background in the presence of a spatially isotropic dielectric medium was studied along with the critical character of the GW effect on the amplification of the electrodynamical response, (here we are using the word “critical” in analogy to phase transitions terminology). It was also shown that one can consider this sort of models in the framework of nonlinear GW fields [5].

One can now go a step further and study the critical behaviour of an electrodynamical system in a GW background for a wider class of models other than those considered in [5]. We discuss in a covariant formalism a generalized problem of the singular behaviour of electrodynamical systems evolving in a GW background. The paper is composed as follows. In section 2 the phenomenological representation of the master equations for electrodynamics of continuous media is given for arbitrary gravitational background fields and then considered for four particular models, namely, pure vacuum, curvature induced corrections in vacuum, spatially isotropic medium, and curvature induced corrections in spatially isotropic medium. In section 3 exact solutions of Maxwell equations in a GW background (specifically, a pp-wave background) are obtained and discussed for the four mentioned examples. Attention is given to the singular behaviour problem. In section 4 we present our conclusions.

2 Master equations and constitutive relationships

2.1 Covariant decription of dielectric and magnetic properties of continuous media and vacuum

The master equations of covariant electrodynamics of continuous media are the following [6, 7]:

$$\nabla_k H^{ik} = -\frac{4\pi}{c} I^i, \quad (1)$$

$$\nabla_k F^{*ik} = 0, \quad (2)$$

where H^{ik} is the electric and magnetic induction tensor, ∇_k is the covariant derivative, I^i is the current four-vector for free charges, F^{*ik} is the tensor dual to the Maxwell tensor F_{mn} . The definition of duality for a generic tensor T^{ik} is

$$T^{*ik} = \frac{1}{2\sqrt{-g}}\epsilon^{ikls}T_{ls}, \quad (3)$$

where $\frac{1}{\sqrt{-g}}\epsilon^{ikls}$ is the Levi-Civita tensor and ϵ^{ikls} is the completely anti-symmetric symbol with $\epsilon^{0123} = -\epsilon_{0123} = 1$. As usually, we have

$$F_{ik} = \nabla_i A_k - \nabla_k A_i, \quad (4)$$

where A_i is the four-vector potential. With this definition equation (2) is trivially satisfied.

The set of equations (1) and (2) of covariant electrodynamics of continuous media should be completed by formulating consistently the constitutive equations [6, 7, 8], linking the electric and magnetic induction tensor with the Maxwell tensor. The simplest relation between these tensors is linear and has the following form:

$$H^{ik} = C^{ikmn}F_{mn}, \quad (5)$$

where C^{ikmn} is called the material tensor, describing the properties of the linear response and containing the information about dielectric and magnetic permeabilities, as well as about the magneto-electric coefficients [6, 7] (see, also, [8] for other details). This tensor has the well-known index-transposition properties:

$$C^{ikmn} = C^{mnik} = -C^{kimn} = -C^{iknm}. \quad (6)$$

Due to these symmetries the C^{ikmn} tensor has 21 independent components.

The physical interpretation of the components of the material tensor C^{ikmn} is simplest in the frame of reference in which the medium is at rest. Thus, the time-like four-vector U^i of the medium is a necessary element of the theory. When the medium is present one has to include the U^i vector into the constitutive equations. Note, however, that in vacuum there is no medium velocity U^i by definition, but one may still use the four-velocity of an arbitrary observer.

The constitutive equations in the form (5) are phenomenological. This phenomenological approach is appropriate to describe several types of media, including the vacuum, which is usually considered as a sort of medium in the framework of covariant electrodynamics of continuous media.

2.2 Decomposition of the material tensor C^{ikmn} : the general case

An interpretation of the 21 independent components of the material tensor C^{ikmn} can be given through relationships between the four-vectors electric induction D^i and magnetic field H^i , on one hand, and the four-vectors electric field E^i and the

magnetic induction B^i on the other hand. In order to do this one defines the vectors D^i , H^i , E^i and B^i by the following formulae [7]:

$$D^i = H^{ik}U_k, \quad H^i = H^{*ik}U_k, \quad E^i = F^{ik}U_k, \quad B^i = F^{*ik}U_k. \quad (7)$$

These vectors are spacelike lying in a hypersurface orthogonal to the velocity four-vector U^i ,

$$D^i U_i = 0 = E^i U_i, \quad H^i U_i = 0 = B^i U_i. \quad (8)$$

For U^i timelike and normalized to unity, one can through formulae (7) and (8) represent the Maxwell tensor in the form

$$F_{mn} = \delta_{mn}^{pq} E_p U_q - \epsilon_{mnls} \sqrt{-g} B^l U^s = E_m U_n - E_n U_m - \eta_{mnl} B^l. \quad (9)$$

where δ_{mn}^{ik} is the generalized 4-indices δ -Kronecker tensor and η_{mnl} is an anti-symmetric tensor orthogonal to U^i defined as

$$\eta_{mnl} \equiv \sqrt{-g} \epsilon_{mnls} U^s, \quad \eta^{ikl} \equiv \frac{1}{\sqrt{-g}} \epsilon^{ikls} U_s. \quad (10)$$

The generalized 6-indices δ -Kronecker tensor and the spacelike tensor η^{ikl} , both constructed from the Levi-Civita tensor, are the standard tools for the decomposition of the material tensor C^{ikmn} . They are connected by the useful identity

$$-\eta^{ikp} \eta_{mnp} = \delta_{mns}^{ikl} U_l U^s = \Delta_m^i \Delta_n^k - \Delta_n^i \Delta_m^k, \quad (11)$$

where the symmetrical projection tensor Δ^{ik} is defined as

$$\Delta^{ik} = g^{ik} - U^i U^k. \quad (12)$$

It allows to project a generic tensor to a hypersurface orthogonal to the four-velocity vector U^i . Upon contraction, equation (11) yields another identity

$$\frac{1}{2} \eta^{ikl} \eta_{klm} = -\delta_{ms}^{il} U_l U^s = -\Delta_m^i. \quad (13)$$

We can now decompose C^{ikmn} uniquely as

$$C^{ikmn} = \frac{1}{2} \left(\varepsilon^{im} U^k U^n - \varepsilon^{in} U^k U^m + \varepsilon^{kn} U^i U^m - \varepsilon^{km} U^i U^n \right) + \frac{1}{2} \left[-\eta^{ikl} (\mu^{-1})_{ls} \eta^{mns} + \eta^{ikl} (U^m \nu_l^{\cdot n} - U^n \nu_l^{\cdot m}) + \eta^{lmn} (U^i \nu_l^{\cdot k} - U^k \nu_l^{\cdot i}) \right]. \quad (14)$$

Here ε^{im} , $(\mu^{-1})_{pq}$ and $\nu_p^{\cdot m}$ are defined as

$$\varepsilon^{im} = 2C^{ikmn} U_k U_n, \quad (\mu^{-1})_{pq} = -\frac{1}{2} \eta_{pik} C^{ikmn} \eta_{mnq}, \quad (15)$$

$$\nu_p^{\cdot m} = \eta_{pik} C^{ikmn} U_n = U_k C^{mkln} \eta_{lnp}. \quad (16)$$

The tensors ε_{ik} and $(\mu^{-1})_{ik}$ are symmetric, but $\nu_l{}^k$ is in general non-symmetric. The dot denotes the position of the second index when lowered. These three tensors are spacelike, i.e., they are orthogonal to U^i ,

$$\varepsilon_{ik}U^k = 0, \quad (\mu^{-1})_{ik}U^k = 0, \quad \nu_l{}^k U^l = 0 = \nu_l{}^k U_k. \quad (17)$$

Calculating D^i and H_i given in (7), and using the decomposition (5) and (14), we obtain the standard linear relationships

$$D^i = \varepsilon^{im}E_m - B^l \nu_l{}^i, \quad H_i = \nu_i{}^m E_m + (\mu^{-1})_{im} B^m. \quad (18)$$

Thus, the tensors ε^{im} , μ_{pq} and $\nu_p{}^m$ are the four-dimensional analogues of the dielectric permeability tensor, the magnetic permeability tensor and the magneto-electric coefficient tensor, respectively. One sees, that the 21 components of C^{ikmn} can be divided into the 6 components of ε^{im} , the 6 components of $(\mu^{-1})_{pq}$ and the 9 components of $\nu_p{}^m$. If the magneto-electric coefficient tensor $\nu_i{}^m$ is equal to zero, we have a special case of a so-called permeable medium and the C^{iklm} tensor is characterized by 12 components only [9].

We now study four models where this decomposition takes place.

2.3 Decomposition of the material tensor C^{ikmn} in vacuum; its dielectric properties in the presence of a gravitational field

2.3.1 The first model: pure vacuum

As a first model we consider the vacuum case in a generic curved background. One can ask what are the coefficients C^{ikmn} for this case. For vacuum electrodynamics one has [6]

$$H^{ik} = F^{ik} = g^{im}g^{kn}F_{mn}. \quad (19)$$

Thus, from (5), the material tensor C^{ikmn} in vacuum must be the following

$$C_{(\text{vac})}^{ikmn} = \frac{1}{2} (g^{im}g^{kn} - g^{in}g^{km}). \quad (20)$$

We can see explicitly, that in vacuum the U^i vector does not appear in C^{ikmn} . From equations (15)-(16) we get

$$\varepsilon^{ik} = (\mu^{-1})^{ik} = \Delta^{ik}, \quad \nu_i{}^k = 0. \quad (21)$$

From (21) we see that the tensors ε^{ik} and $(\mu^{-1})^{ik}$ depend on the four-velocity, in contrast to the $C_{(\text{vac})}^{ikmn}$ tensor (20). Thus, the vacuum in a gravitational field should be seen as a specific sort of medium by specifying a four-velocity to it [6, 10].

2.3.2 The second model: curvature induced corrections in vacuum

As a second model we consider the one-loop corrections to quantum electrodynamics in a curved vacuum background [11]. The relationship between induction tensor and Maxwell tensor has the form [11]

$$H^{ik} = F^{ik}(1 + q_1 R) + q_2(R_m^i F^{mk} - R_m^k F^{mi}) + q_3 R^{ikmn} F_{mn}, \quad (22)$$

where R^{ikmn} is the Riemann tensor, R^{ik} is the Ricci tensor, R is the scalar curvature of the spacetime, and the coefficients

$$q_1 = -\frac{\alpha\lambda_e^2}{180\pi}, \quad q_2 = \frac{13\alpha\lambda_e^2}{180\pi}, \quad q_3 = -\frac{\alpha\lambda_e^2}{90\pi} \quad (23)$$

contain the fine structure constant α and Compton wavelength of the electron λ_e . From (22) one sees that the interaction between the electromagnetic field and curvature produces an electric and magnetic polarization of the vacuum. Using (5), we can conclude that the material tensor is now given by

$$C^{ikmn} = C_{(\text{vac})}^{ikmn} + C_{(\text{corr})}^{ikmn}, \quad (24)$$

where,

$$C_{(\text{corr})}^{ikmn} = \frac{1}{2}q_1 R (g^{im}g^{kn} - g^{in}g^{km}) + \frac{1}{2}q_2 (R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in}) + q_3 R^{ikmn}. \quad (25)$$

Note, that $C_{(\text{corr})}^{ikmn}$ is linear in the Riemann, Ricci and scalar curvatures and does not contain the four-velocity vector U^i .

2.4 Decomposition of C^{ikmn} in a spatially isotropic medium

2.4.1 The third model: pure spatially isotropic medium in a gravitational field

As a third model we consider a transparent spatially isotropic medium. From the phenomenological point of view a spatially isotropic medium may be described by two scalars only: ε - the dielectric permeability and μ - the magnetic permeability.

Note the similarity of a spatially isotropic medium with its two scalars ε and μ , with the theory of elasticity also characterized by two scalars, the Lamé coefficients λ and μ . Thus, keeping this analogy, and recalling that the symmetric tensor of elastic modulus E^{iklm} can be decomposed as $E^{iklm} = \lambda\Delta^{ik}\Delta^{lm} + \mu(\Delta^{il}\Delta^{km} + \Delta^{im}\Delta^{kl})$, one can also decompose the anti-symmetric tensor C^{iklm} using two tensorial quantities, G^{ikmn} and Δ^{ikmn} , defined by

$$G^{ikmn} \equiv \frac{1}{2} (g^{im}g^{kn} - g^{in}g^{km}) \quad (26)$$

and Δ^{ikmn}

$$\Delta^{ikmn} \equiv \frac{1}{2} \left(\Delta^{im} \Delta^{kn} - \Delta^{in} \Delta^{km} \right). \quad (27)$$

These tensors form the basis for the decomposition of the anti-symmetric material tensor in a spatially isotropic medium. We can then write the following linear combination

$$C_{(\text{iso})}^{ikmn} \equiv \frac{1}{\mu} \Delta^{ikmn} + \varepsilon \left(G^{ikmn} - \Delta^{ikmn} \right), \quad (28)$$

or in detailed form,

$$C_{(\text{iso})}^{ikmn} = \frac{1}{2\mu} \left(g^{im} g^{kn} - g^{in} g^{km} \right) + \left(\frac{\varepsilon\mu-1}{2\mu} \right) \left(g^{im} U^k U^n - g^{in} U^k U^m + g^{kn} U^i U^m - g^{km} U^i U^n \right), \quad (29)$$

where the coefficients in this linear combination were choosen to recover the standard definitions in three dimensions, i.e.,

$$D^i = \varepsilon E^i, \quad H_i = \frac{1}{\mu} B_i, \quad \varepsilon^{ik} = \varepsilon \Delta^{ik}, \quad (\mu^{-1})^{ik} = \frac{1}{\mu} \Delta^{ik}, \quad \nu_i^{\cdot k} = 0. \quad (30)$$

If we put ε and μ equal to unity, equation (29) yields, the material tensor in vacuum (20), as it should.

2.4.2 The fourth model: curvature induced anisotropic corrections in a spatially isotropic medium

As before, through a phenomenological decomposition, we find the anisotropic contributions to C^{ikmn} from the Riemann tensor, the Ricci tensor and the scalar curvature. These contributions give

$$C_{(\text{aniso})}^{ikmn} = C_{(\text{iso})}^{ikmn} + C_{(\text{corr})}^{ikmn}, \quad (31)$$

where now,

$$\begin{aligned} C_{(\text{corr})}^{ikmn} = & (H_1 R + Q_1 R_{pq} U^p U^q) \left(g^{im} g^{kn} - g^{in} g^{km} \right) + \\ & + \left(\bar{H}_1 R + \bar{Q}_1 R_{pq} U^p U^q \right) \left(g^{im} U^k U^n - g^{in} U^k U^m + g^{kn} U^i U^m - g^{km} U^i U^n \right) + \\ & + Q_2 \left(R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in} \right) + \\ & + \bar{Q}_2 \left(R^{im} U^k U^n - R^{in} U^k U^m + R^{kn} U^i U^m - R^{km} U^i U^n \right) + \\ & + \hat{Q}_2 U^l \left(R_l^{(i} U^m) U^k U^n - R_l^{(i} U^n) U^k U^m + R_l^{(k} U^n) U^i U^m - R_l^{(k} U^m) U^i U^n \right) + \\ & + Q_3 R^{ikmn} + \\ & + \bar{Q}_3 U_p U_q \left(R^{ipmq} g^{kn} - R^{ipnq} g^{km} + R^{kp nq} g^{im} - R^{kpmq} g^{in} \right) + \\ & + \hat{Q}_3 U_p U_q \left(R^{ipmq} U^k U^n - R^{ipnq} U^k U^m + R^{kp nq} U^i U^m - R^{kpmq} U^i U^n \right), \end{aligned} \quad (32)$$

where, $(i m)$ denotes symmetrization in those indices, and the H_1 and \bar{H}_1 , $Q_1, Q_2, Q_3, \bar{Q}_1, \bar{Q}_2, \bar{Q}_3$ and \hat{Q}_2, \hat{Q}_3 coefficients are phenomenological parameters for the medium,

which appear in this formalism in much the same way as q_1, q_2, q_3 have appeared in the vacuum case. However, we stress that contrary to the vacuum case where q_1, q_2 , and q_3 can be found from within vacuum quantum electrodynamics [11], the coefficients H_1 , etc, cannot be directly extracted, since there is no developed self-consistent quantum electrodynamics in continuous media in curved spacetimes.

The full material tensor (31), $C_{(\text{aniso})}^{ikmn} = C_{(\text{iso})}^{ikmn} + C_{(\text{corr})}^{ikmn}$, containing contributions from (29) and (32), covers all three previous examples (20), (25), and (29), on choosing the coefficients in the decompositions (29) and (32) in an appropriate way. In this sense it is the general construction to the four mentioned examples given above. For instance, for $\varepsilon = \mu = 1$ we may, if we wish, restrict to the case where the coefficients $\bar{H}_1, Q_1, \bar{Q}_1, \bar{Q}_2, \bar{Q}_3, \hat{Q}_2, \hat{Q}_3$ in (32) disappear. This could be achieved by putting these coefficients proportional to $(\varepsilon\mu - 1)^\Gamma$, ($\Gamma > 0$). Note, however, that from the phenomenological point of view this requirement is not compulsory.

In order to be complete, we mention that the material tensor C^{iklm} we have been using can be enlarged to contain other admissible non-minimal coupling terms [12]. Indeed, the most general material tensor χ^{iklm} , without curvature corrections, has the generic form $\chi = f(x)\chi^0 + \alpha(x)[\epsilon]$, where we have suppressed the indices, $[\epsilon]$ here means tensors constructed from an odd number of the Levi-Civita tensor, and x stands for spacetime coordinates. The scalar function $\alpha(x)$ is an Abelian axion field which appears in some particle theories with CP-violation [8, 13]. Adding curvature corrections leads to a non-minimal coupling contribution of duals of the curvature tensors, R^{*iklm} for instance, to the tensor χ^{iklm} [12]. This approach would lead to an interaction term between the axions and the curvature, but we do not investigate these models here.

3 Exact solutions of Maxwell equations

3.1 Decomposition of the material tensor C^{ikmn} for a pp-wave gravitational background and its dielectric properties

The previous sections contain the decomposition of the material tensor for an arbitrary spacetime metric. It is of interest to study the tensor C^{ikmn} in a particular background. Here we will consider as the spacetime background the exact pp-wave solution of Einstein's equations in vacuum described by the metric [10, 14]

$$ds^2 = 2dudv - L^2 \left[e^{2\beta} (dx^2)^2 + e^{-2\beta} (dx^3)^2 \right], \quad (33)$$

where

$$u = \frac{ct - x^1}{\sqrt{2}}, \quad v = \frac{ct + x^1}{\sqrt{2}} \quad (34)$$

are the retarded and the advanced time, respectively. The functions L and β depend on u only.

The pp-wave metric (33) possesses G_5 as the symmetry group [14] and admits the following set of Killing vector fields:

$$\begin{aligned}\xi_{(v)}^i &= \delta_v^i, & \xi_{(2)}^i &= \delta_2^i, & \xi_{(3)}^i &= \delta_3^i, \\ \xi_{(4)}^i &= x^2 \delta_v^i - \delta_2^i \int g^{22}(u) du, & \xi_{(5)}^i &= x^3 \delta_v^i - \delta_3^i \int g^{33}(u) du,\end{aligned}\quad (35)$$

of which the first three, $\xi_{(v)}$, $\xi_{(2)}$, and $\xi_{(3)}$, form a G_3 Abelian subgroup of G_5 . Here $g^{\alpha\beta}(u)$ ($\alpha, \beta = 2, 3$) are the contravariant components of the metric tensor. The vector $\xi_{(v)}^i$ is isotropic, covariantly constant and orthogonal to the other four ones, i.e.,

$$\nabla_k \xi_{(v)}^i = 0, \quad g_{ik} \xi_{(v)}^i \xi_{(j)}^k = 0. \quad (36)$$

The two functions $L(u)$ and $\beta(u)$ are coupled by the equation:

$$L'' + L(\beta')^2 = 0. \quad (37)$$

One can assume $\beta(u)$ as an arbitrary function of u and then solve for L . The curvature tensor has two non-zero components:

$$-R_{.u2u}^2 = R_{.u3u}^3 = L^{-2} [L^2 \beta']'. \quad (38)$$

The Ricci tensor R_{ik} and the curvature scalar R are equal to zero.

For a medium (or an observer) at rest in the chosen frame of reference one has

$$U^i = (\delta_u^i + \delta_v^i) \frac{1}{\sqrt{2}}. \quad (39)$$

Then one can show that for the pp-wave background, described by equations (33)-(38), one obtains for a spatially isotropic medium with curvature induced corrections (see (31)-(32)), the following dielectric permeability, magnetic permeability and magneto-electric coefficients, respectively:

$$\begin{aligned}\varepsilon^{im} &= \varepsilon \Delta^{im} + 2(Q_3 + \bar{Q}_3 + \hat{Q}_3) R^{ikmn} U_k U_n, \\ (\mu^{-1})_{pq} &= \frac{1}{\mu} \Delta_{pq} - \frac{1}{2} Q_3 \eta_{pik} R^{ikmn} \eta_{mnq} - 2\bar{Q}_3 R_{plqs} U^l U^s, \\ \nu_p^m &= Q_3 \eta_{pik} R^{ikmn} U_n.\end{aligned}\quad (40)$$

Explicitly we find,

$$\begin{aligned}\varepsilon^{uu} &= \varepsilon^{vv} = -\varepsilon^{uv} = \frac{\varepsilon}{2}, & (\mu^{-1})_{uu} &= (\mu^{-1})_{vv} = -(\mu^{-1})_{uv} = \frac{1}{2\mu}, \\ \varepsilon^{u\alpha} &= (\mu^{-1})_{u\alpha} = \varepsilon^{v\alpha} = (\mu^{-1})_{v\alpha} = 0, & \varepsilon^{23} &= (\mu^{-1})_{23} = 0, \\ \varepsilon^{22} &= g^{22} [\varepsilon + (Q_3 + \bar{Q}_3 + \hat{Q}_3) R_{.u2u}^2], & \varepsilon^{33} &= g^{33} [\varepsilon - (Q_3 + \bar{Q}_3 + \hat{Q}_3) R_{.u2u}^2], \\ (\mu^{-1})^{22} &= g^{22} [\frac{1}{\mu} + (Q_3 - \bar{Q}_3) R_{.u2u}^2], & (\mu^{-1})^{33} &= g^{33} [\frac{1}{\mu} - (Q_3 - \bar{Q}_3) R_{.u2u}^2], \\ \nu_{uu} &= \nu_{uv} = \nu_{vu} = \nu_{vv} = \nu_{u\alpha} = \nu_{\alpha u} = \nu_{v\alpha} = \nu_{\alpha v} = \nu_{22} = \nu_{33} = 0, \\ \nu_{23} &= \nu_{32} = -Q_3 L^2 R_{.u2u}^2.\end{aligned}\quad (41)$$

One sees, that the pp-wave field induces an anisotropy in the dielectric properties of the initially spatially isotropic medium. These anisotropies appear in the plane $x^2 O x^3$ only, parallel to the gravitational wave front.

One important question arises: do these modifications in the dielectric properties of the media produce changes in the electromagnetic field structure and in its properties? In order to answer this question we consider now some exact solutions of Maxwell equations.

3.2 States inheriting the symmetry of the GW background and the reduction of Maxwell equations

Searching for exact solutions of Maxwell equations (1)-(2), it is advisable to fully use the inherited spacetime symmetries. Thus, we suggest first, that, for negative retarded time ($u \leq 0$) before the appearance of the gravitational wave (at $u = 0$) the dielectric medium is infinite, homogeneous and static. In other words, the tensor C^{ikmn} can be put equal to a constant at $u \leq 0$.

For $u > 0$, i.e., the GW already permeates the medium, the tensor C^{ikmn} contains the metric $g_{ik}(u)$ and the Riemann tensor $R^{ikmn}(u)$ (see, e.g., (32)). Therefore, C^{ikmn} depends on the retarded time only. This idea can be formulated in a covariant way.

Call Ψ an arbitrary macroscopic function of the state of the electrodynamical system (material tensor, Maxwell tensor, induction tensor, etc.), and, as before, let $\xi_{(a)}^i$ be the Killing vectors belonging to the Abelian subgroup G_3 of the total G_5 group of symmetry of the GW space-time. Now, if the electrodynamical quantities are to inherit the spacetime symmetries one must impose, $\mathcal{L}_{\xi_{(a)}} \Psi = 0$, where $\mathcal{L}_{\xi_{(a)}}$ stands for the Lie derivative along the $\xi_{(a)}^i$ vector. This zero Lie-derivative condition on the Maxwell and induction tensors, i.e., $\mathcal{L}_{\xi_{(a)}} F_{mn} = 0$ and $\mathcal{L}_{\xi_{(a)}} H_{mn} = 0$, implies that they depend on the variable u only, $F_{mn} = F_{mn}(u)$ and $H_{mn} = H_{mn}(u)$.

Thus, the second subsystem of Maxwell equations (2) can be put in the form

$$\frac{1}{2L^2} \frac{d}{du} (\epsilon^{iuls} F_{ls}) = 0. \quad (42)$$

On integrating equation (42) we obtain,

$$F_{v\alpha}(u) = F_{v\alpha}(0) = \text{const}, \quad F_{\alpha\beta}(u) = F_{\alpha\beta}(0) = \text{const}. \quad (43)$$

Analogously, the first subsystem of Maxwell equations (1) yields

$$L^2 C^{iumn}(u) F_{mn}(u) = C^{iumn}(0) F_{mn}(0) = \text{const}. \quad (44)$$

Only three equations of (44) are non-trivial (the equations for $i = v, 2, 3$). The three functions $F_{u\alpha}$ and F_{uv} are unknown, since $F_{v\alpha}$ and $F_{\alpha\beta}$ are constant (see (43)) and can be expressed in terms of initial data on the GW front. Thus, we obtain the following reduced algebraic system, containing three equations for the three unknown functions

$$C^{vvuv}(u) F_{uv}(u) + C^{vu\alpha u}(u) F_{u\alpha}(u) = Z^v(u), \quad (45)$$

$$C^{\gamma uvu}(u) F_{uv}(u) + C^{\gamma u\alpha u}(u) F_{u\alpha}(u) = Z^\gamma(u), \quad (46)$$

where

$$Z^v(u) = \frac{1}{L^2} C^{vvuv}(0) F_{uv}(0) + \frac{1}{L^2} C^{vu\alpha u}(0) F_{u\alpha}(0) + \\ + F_{v\alpha}(0) \left(C^{vvu\alpha}(u) - \frac{1}{L^2} C^{vvu\alpha}(0) \right) + \frac{1}{2} F_{\alpha\beta}(0) \left(C^{vu\alpha\beta}(u) - \frac{1}{L^2} C^{vu\alpha\beta}(0) \right), \quad (47)$$

and,

$$Z^\gamma(u) = \frac{1}{L^2} C^{\gamma uvu}(0) F_{uv}(0) + \frac{1}{L^2} C^{\gamma u\alpha u}(0) F_{u\alpha}(0) + \\ + F_{v\alpha}(0) \left(C^{\gamma uv\alpha}(u) - \frac{1}{L^2} C^{\gamma uv\alpha}(0) \right) + \frac{1}{2} F_{\alpha\beta}(0) \left(C^{\gamma u\alpha\beta}(u) - \frac{1}{L^2} C^{\gamma u\alpha\beta}(0) \right), \quad (48)$$

are known functions of the retarded time u .

This system of three equations (45)-(46) can be solved for generic material tensor (14) by the standard procedure involving Cramer's determinant, as in a linear algebra system of equations.

This system has an unique solution if the determinant \mathcal{D} of the 3×3 matrix $C^{iuku}(u)$ ($i, k = v, 2, 3$) is not identically equal to zero. For such a unique solution, the components F_{uk} of the Maxwell tensor contain the determinant \mathcal{D} in their denominators.

We introduce the term “singular behaviour” in the description of the behaviour of the electromagnetic field, when the determinant \mathcal{D} is not equal to zero identically, but close to zero. In principle, using equation (45)-(46) one can formulate explicitly this requirement in a general case for a generic anisotropic medium. Nevertheless, to clarify the physical meaning of the singularities, we consider the four particular previous examples of the C^{iklm} tensor decomposition.

3.3 Exact solutions in vacuum

3.3.1 The first model: pure vacuum

The refractive index is defined by $n^2 \equiv \varepsilon\mu$. For pure vacuum $n^2 = 1$. When $n^2 = 1$, Cramer's determinant for the system (45)-(46) is equal to zero. Then, equation (45) gives

$$F_{uv}(u) = \frac{1}{L^2} F_{uv}(0), \quad (49)$$

and equation (46) yields

$$F_{v\alpha}(0) = 0. \quad (50)$$

Due to the condition (50), the solution of the system of equations (46) exists if and only if $F_{v\alpha}(u) = 0$. One also has that F_{23} and $F_{u\alpha}$ are arbitrary functions of u , $F_{23}(u)$ and $F_{u\alpha}(u)$, since they do not appear in the set of equations. This solution describes a static longitudinal electric and magnetic field (longitudinal with respect to the direction of the GW propagation). However, the transversal components of the field represent a travelling wave. Thus, generally, when $n^2 = 1$, the required solution of Maxwell equations, depending on the retarded time u only, does not exist. In other words, the problem of the response of the electrodynamic system in vacuum (i.e., without dielectric material) is a subject of special consideration, which we do now.

Let us focus on the evolution of a magnetic field (we do not treat here an electric field, since it yields some conceptual problems). In the absence of GWs the magnetic field is static and homogeneous, i.e., it means is constant throughout space. Let us call the three components of this constant field as H^1 , H^2 , and H^3 . In the GW background metric (33) we have now to find a solution of the Maxwell equations depending on the four coordinates (u, v, x^2, x^3) , since there is no solution depending on the retarded time u alone. In order to satisfy the initial conditions, i.e., in the absence of a GW the magnetic field is equal to an arbitrary constant vector, one can choose, for $u \leq 0$, the components of the electromagnetic potential four-vector A_i depending linearly on the coordinates (u, v, x^2, x^3) :

$$A_u = A_v = 0, \quad A_2 = \frac{1}{\sqrt{2}}H^3(u-v) + \frac{1}{2}H^1x^3, \quad A_3 = \frac{1}{\sqrt{2}}H^2(v-u) - \frac{1}{2}H^1x^2. \quad (51)$$

The Maxwell equations admit the solution in which A_u and A_v are not changed by the GW, and the transversal components of the four-vector of potential satisfy the equations [15]

$$\hat{D}A_2 = 2\beta' \partial_v A_2, \quad \hat{D}A_3 = -2\beta' \partial_v A_3, \quad (52)$$

where $\hat{D} \equiv 2\partial_u \partial_v + g^{\alpha\beta} \partial_\alpha \partial_\beta$. It is easy to check that the formulae

$$\begin{aligned} F_{uv} &\equiv F_{01} = 0, & F_{32} &= H^1, \\ F_{v2} &= -\frac{1}{\sqrt{2}}H^3e^\beta, & F_{v3} &= \frac{1}{\sqrt{2}}H^2e^{-\beta}, \\ F_{u2} &= -F_{v2} + \beta' e^\beta M_2, & F_{u3} &= -F_{v3} - \beta' e^{-\beta} M_3, \end{aligned} \quad (53)$$

where M_2 and M_3 are defined by

$$\begin{aligned} M_2 &\equiv \frac{1}{\sqrt{2}}H^3(u-v) + \frac{1}{2}H^1x^3e^{-\beta}, \\ M_3 &\equiv \frac{1}{\sqrt{2}}H^2(v-u) - \frac{1}{2}H^1x^2e^\beta, \end{aligned} \quad (54)$$

represent the solution of equations (1)-(2) in vacuum with initial conditions (51).

Formulae (53)-(54) recover the known results of [1, 2, 3], when we reduce the nonlinear GW background to the case of a weak GW. It is important to emphasize that the gravitationally induced electric field, whose components are,

$$E_2 = -\frac{1}{\sqrt{2}}\beta' e^\beta M_2, \quad E_3 = \frac{1}{\sqrt{2}}\beta' e^{-\beta} M_3, \quad (55)$$

grows linearly as a function of the coordinates (v, x^2, x^3) .

3.3.2 The second model: curvature induced corrections in vacuum

From equations (45) one obtains equation (49), as in the previous case. Then equation (46) also coincides with equation (50) for pure vacuum because the curvature induced corrections for the material tensor (see equation (25)) do not give any contribution to C^{iuku} tensor in equations (45)-(46). As in the previous case there is no solution depending on u only. For a different solution see [15].

3.4 Exact solutions in a spatially isotropic medium

3.4.1 The third model: pure spatially isotropic medium in a gravitational wave field

Consider now a spatially isotropic medium with $n^2 \neq 1$. Using formula (29) for the C^{ikmn} coefficients and equation (39) for the four-velocity of the medium, one can extract that equation (45) also gives equation (49). The remaining pair of equations (46) takes the form

$$(n^2 - 1)g^{\gamma\alpha}F_{u\alpha}(u) = (n^2 - 1)\frac{\eta^{\gamma\alpha}}{L^2}F_{u\alpha}(0) + (n^2 + 1)F_{v\alpha}(0)\left(g^{\gamma\alpha} - \frac{1}{L^2}\eta^{\gamma\alpha}\right), \quad (56)$$

where η_{ik} is the Minkowski metric.

For $n^2 \neq 1$, the solution of the system (56) is unique and equal to

$$\begin{aligned} F_{u2}(u) &= F_{u2}(0) + F_{u2}(0)\left(e^{2\beta} - 1\right) + \left(\frac{n^2+1}{n^2-1}\right)F_{v2}(0)\left(e^{2\beta} - 1\right), \\ F_{u3}(u) &= F_{u3}(0) + F_{u3}(0)\left(e^{-2\beta} - 1\right) + \left(\frac{n^2+1}{n^2-1}\right)F_{v3}(0)\left(e^{-2\beta} - 1\right). \end{aligned} \quad (57)$$

The physical analysis of the solution (49) and (57) is more transparent in terms of the electric and magnetic four-vectors (7):

$$\begin{aligned} E_1(u) &\equiv F_{10} \equiv -F_{uv} = \frac{1}{L^2}E_1(0), \\ B^1(u) &\equiv F_{32} = B^1(0), \\ E_2(u) &\equiv F_{20} \equiv \frac{1}{\sqrt{2}}(F_{2u} + F_{2v}) = E_2(0) + \frac{1}{(n^2-1)}\Pi_2(u), \\ B^2(u) &\equiv F_{13} \equiv \frac{1}{\sqrt{2}}(F_{v3} - F_{u3}) = B^2(0) + \frac{1}{(n^2-1)}\Pi_3(u), \\ E_3(u) &\equiv F_{30} \equiv \frac{1}{\sqrt{2}}(F_{3u} + F_{3v}) = E_3(0) + \frac{1}{(n^2-1)}\Pi_3(u), \\ B^3(u) &\equiv F_{21} \equiv \frac{1}{\sqrt{2}}(F_{u2} - F_{v2}) = B^3(0) - \frac{1}{(n^2-1)}\Pi_2(u), \end{aligned} \quad (58)$$

where

$$\begin{aligned} \Pi_2(u) &= \left(e^{2\beta} - 1\right)[n^2E_2(0) + B^3(0)], \\ \Pi_3(u) &= \left(e^{-2\beta} - 1\right)[n^2E_3(0) - B^2(0)]. \end{aligned} \quad (59)$$

We see that when $n^2 \rightarrow 1$, then the gravitationally induced part of the electric and magnetic fields, i.e., the terms involving $\frac{1}{(n^2-1)}\Pi_\alpha(u)$, go to infinity (we are not taking $n^2E_2(0) = -B^3(0)$, and $n^2E_3(0) = B^2(0)$ simultaneously, which is a rather special case, of no interest here). Thus, we have two distinct situations (i) if we take first the limit of $\beta \rightarrow 0$ and then $n^2 \rightarrow 1$ we obtain that the gravitational induced part of the Maxwell tensor F_{ik} is zero, i.e., the double limit: $\lim_{n^2 \rightarrow 1} \lim_{\beta \rightarrow 0} \{F_{ik}\} = 0$, where $\{F_{ik}\}$ denotes the gravitational induced part of the Maxwell tensor F_{ik} ; (ii) if we take first the limit of $n^2 \rightarrow 1$ and then $\beta \rightarrow 0$ we obtain that the gravitational induced part of the Maxwell tensor F_{ik} is infinite, i.e., the double limit $\lim_{\beta \rightarrow 0} \lim_{n^2 \rightarrow 1} \{F_{ik}\} = \infty$. Since these limits do not coincide, we can speak of a critical behaviour of the electromagnetic field near the singular point $n^2 = 1$. In the absence of the GW such a problem does not arise.

3.4.2 The fourth model: curvature induced anisotropic corrections in a spatially isotropic medium

When the tensor of linear response C^{ikmn} is given by the formulae (31)-(32), then the longitudinal component of the electric field coincides with (49), but the remaining two-component subsystem of equations (46) takes the form

$$\begin{aligned} \left[(n^2 - 1)g^{\gamma\alpha} + \mu\hat{Q}_3 R^{\gamma\nu\alpha\nu} \right] F_{u\alpha}(u) &= (n^2 + 1)F_{v\alpha}(0) \left(g^{\gamma\alpha} - \frac{1}{L^2}\eta^{\gamma\alpha} \right) + \\ &+ (n^2 - 1)\frac{\eta^{\gamma\alpha}}{L^2}F_{u\alpha}(0) - \mu F_{v\alpha}(0)R^{\gamma\nu\alpha\nu}(\hat{Q}_3 + 2\bar{Q}_3). \end{aligned} \quad (60)$$

In this case Cramer's determinant \mathcal{D} of the system (60) is equal to

$$\mathcal{D} = \frac{1}{L^4} \left[(n^2 - 1) + \mu\hat{Q}_3 R_{u2u}^2 \right] \left[(n^2 - 1) + \mu\hat{Q}_3 R_{u3u}^3 \right]. \quad (61)$$

Recalling that $R_{u2u}^2 = -R_{u3u}^3$, we find

$$\mathcal{D} = \frac{1}{L^4} \left[(n^2 - 1)^2 - \mu^2 \hat{Q}_3^2 (R_{u2u}^2)^2 \right], \quad (62)$$

and

$$\begin{aligned} F_{u2}(u) &= \left[1 + \frac{\mu\hat{Q}_3}{(n^2 - 1)} R_{u2u}^2 \right]^{-1} \left\{ F_{v2}(0) \frac{1}{(n^2 - 1)} \left[(n^2 + 1) (1 - e^{2\beta}) \right. \right. \\ &\quad \left. \left. - \mu R_{u2u}^2 (\hat{Q}_3 + 2\bar{Q}_3) \right] + F_{u2}(0) e^{2\beta} \right\}, \\ F_{u3}(u) &= \left[1 + \frac{\mu\hat{Q}_3}{(n^2 - 1)} R_{u3u}^3 \right]^{-1} \left\{ F_{v3}(0) \frac{1}{(n^2 - 1)} \left[(n^2 + 1) (1 - e^{-2\beta}) \right. \right. \\ &\quad \left. \left. - \mu R_{u3u}^3 (\hat{Q}_3 + 2\bar{Q}_2) \right] + F_{u3}(0) e^{-2\beta} \right\}. \end{aligned} \quad (63)$$

There is a singularity when $\mathcal{D} \rightarrow 0$. For instance, this can happen, when n^2 is near one and the quantity $\mu^2 \hat{Q}_3^2 (R_{u2u}^2)^2$ is equal to the small difference $(n^2 - 1)^2$. Since the Riemann tensor can periodically change its value, due to the periodic oscillations of the GW, one could observe a periodic increasing of one of the two component of the electric and magnetic fields. This type of singularities is very similar to the behaviour of the previous third model. The novelty here, is that periodic variations of the curvature tensor produce a natural approach to the critical point at $\mathcal{D} = 0$, which repeats itself periodically. In principle, from the phenomenological point of view, we can consider a new sort of medium with $n^2 = 1$, $\mu = 1$, and $\hat{Q}_3 \neq 0$. Then $\mathcal{D} = -\hat{Q}_3^2 (R_{u2u}^2)^2 / L^4$. From equation (32), we see that for $\hat{Q}_3 \neq 0$ the material tensor depends on the velocity U^i , which is a characteristic of media in general. Therefore, we can call this medium a “quasi-vacuum”.

4 Conclusions

We have shown that there exists a new class of solutions of evolutionary equations for initially static electric and magnetic fields inside a media in a GW background.

The solutions on this class have two important properties: the first property is that they inherit the symmetry of the GW field and depend on the retarded time only. In vacuum there are no such solutions, as we have shown. The second property is connected with the singularities which appear in the electro-magnetic response to the GW. These singularities have a general status and appear not only in a material media with arbitrary spatial symmetry, but also in a “quasi-media”, such as vacuum interacting with curvature. This second property of the solutions gives a clear motivation to search for experimental confirmation of this theoretical predictions, namely the amplification of electric and magnetic fields due to the passage of a GW.

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